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Exact solution for random walks on the triangular lattice with absorbing boundaries

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Abstract
The problem of a random walk on a finite triangular lattice with a single interior source point and zig-zag absorbing boundaries is solved exactly. This problem has been previously considered intractable.

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1. Introduction

The problem of a random walk on a two-dimensional lattice with a single interior source point and absorbing boundaries was first considered by Courant *et al* [1] in 1928 where general properties of the solution were discussed. This problem was solved exactly in 1940 [2] for the case of random walks on a square lattice with rectangular absorbing boundaries. The problem on a triangular lattice with finite absorbing boundaries was considered in 1963 [3] where the exact solution was given for an approximation of the problem using straight boundaries rather than the true zig-zag boundaries of the triangular lattice. Indeed the authors of this paper remarked that 'an explicit solution of the difference equation can hardly be obtained if these boundary conditions are used'. Other variants of the problem on the square lattice have been solved exactly [3, 4], however the problem on the triangular lattice with true zig-zag triangular lattice boundaries has remained unsolved. In this paper we give the exact solution for this problem.

The problem of random walks on finite lattices with absorbing boundaries is fundamental to the theory of stochastic processes [5] and has numerous applications. These include potential theory [6], electrical networks [7], surface diffusion [8] and diffusion-limited aggregation (DLA) [9]. For example, the exact results for the square lattice problem [2] have been used to expedite the growth of large DLA clusters on the square lattice [10] and the formulae derived in this paper could similarly be used on the triangular lattice.

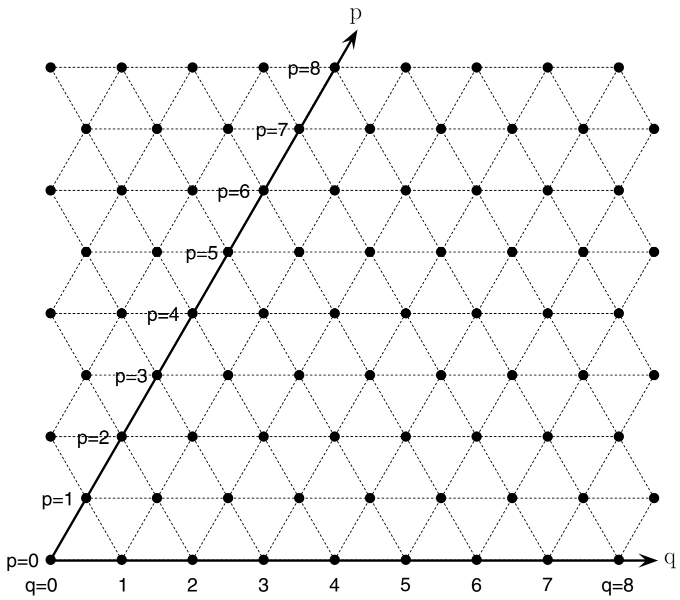

Figure 1. An equi-angular triangular lattice with natural (p, q) coordinates.

2. Field equations, boundary conditions and absorption probabilities

A schematic illustration showing random walk pathways (dashed lines) on an equiangular triangular lattice is shown in figure 1. A natural coordinate system for the triangular lattice is to label the lattice vertices by the intersection points (p, q) of horizontal straight lines $p = 0, 1, 2, \ldots, m+1$ and slanted straight lines $q = 0, 1, 2, \ldots, n+1$—parallel to the p-axis shown in figure 1. In this coordinate system, for a random walk starting at a lattice site (a, b), the expectation that the walk visits a site (p, q) distinct from site (a, b) before exiting at a finite boundary is given by the non-separable difference equation

$$F(p,q) = \tfrac{1}{6}[F(p,q-1) + F(p,q+1) + F(p-1,q) + F(p+1,q) \\ + F(p+1,q-1) + F(p-1,q+1)].$$

In this paper we adopt a different coordinate system, which may at first appear less natural but it has the advantage that the governing equations are separable.

In the coordinate system used here we label the vertices of the triangular lattice by the intersection points (p, q) of horizontal straight lines $p = 0, 1, 2, \ldots, m+1$ and vertical zig-zag lines $q = 0, 1, 2, \ldots, n+1$ (see figure 2). If (p, q) is an interior site distinct from the starting site (a, b) then the expectation that the random walker visits the lattice site (p, q) is given by the coupled homogeneous difference equations

$$F(p,q) = \tfrac{1}{6}[\hat{F}(p-1,q-1) + F(p,q-1) + \hat{F}(p+1,q-1) + \hat{F}(p+1,q) \\ + F(p,q+1) + \hat{F}(p-1,q)] \tag{1}$$

$$\hat{F}(p,q) = \tfrac{1}{6}[F(p-1,q) + \hat{F}(p,q-1) + F(p+1,q) + F(p+1,q+1) \\ + \hat{F}(p,q+1) + F(p-1,q+1)] \tag{2}$$

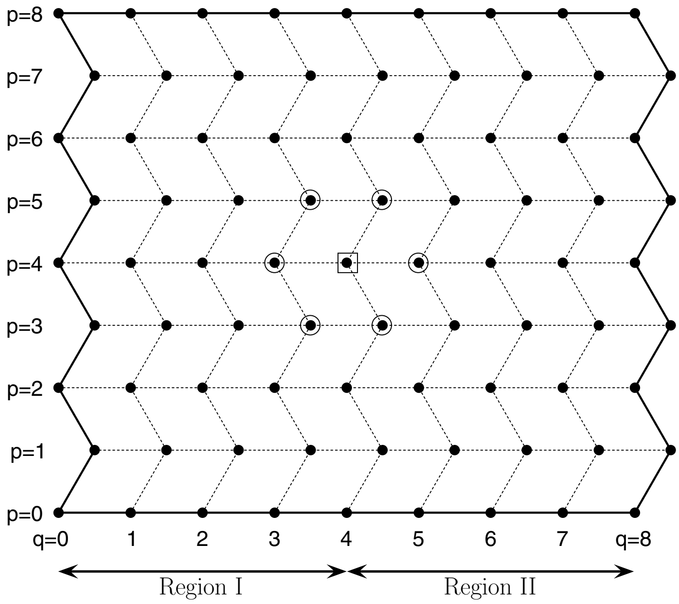

Figure 2. Coordinate system, represented by dashed lines, adopted in this paper for random walks on the triangular lattice with true zig-zag lattice boundaries. In this example $m = 7, n = 7$. Nearest neighbour sites, highlighted with open circles, are shown surrounding a source point, highlighted by an open box, at $a = 4, b = 4$. The absorbing boundary sites are connected by solid lines.

where $F(p, q)$ denotes the field value at an even p coordinate and $\hat{F}(p, q)$ denotes the field value at an odd p coordinate. This separation of even and odd field equations, which is not necessary for the problem on the square lattice, is central to our solution below.

Now consider the equations with the source term at (a, b) and the absorbing boundaries at $p = 0, m + 1$ and $q = 0, n + 1$. In section 3 we give the solution for the case where $m > 1$ is odd. The same methods can be used with slightly altered equations to obtain the solution for m even. The special case of the strip with $m = 1$ has been detailed elsewhere [11]. Following McCrea and Whipple [2] we construct separate solutions, $F_{\mathrm{I}}(p, q)$, $\hat{F}_{\mathrm{I}}(p, q)$ for $q \leqslant b$ and $F_{\mathrm{II}}(p, q)$, $\hat{F}_{\mathrm{II}}(p, q)$ for $q \geqslant b$. The absorbing boundary conditions are thus given by

$$F(0, q) = 0 \tag{3}$$

$$F(m + 1, q) = 0 \tag{4}$$

$$F_{\mathrm{I}}(p, 0) = 0 \tag{5}$$

$$\hat{F}_{\mathrm{I}}(p, 0) = 0 \tag{6}$$

$$F_{\mathrm{II}}(p, n + 1) = 0 \tag{7}$$

$$\hat{F}_{\mathrm{II}}(p, n + 1) = 0. \tag{8}$$

The omission of a subscript I or II in the above indicates that the same equations are satisfied by both F_{I} and F_{II}.

At $q = b$ we have the matching conditions

$$F_{\mathrm{I}}(p, b) = F_{\mathrm{II}}(p, b) \tag{9}$$

$$\hat{F}_{\mathrm{I}}(p, b) = \hat{F}_{\mathrm{II}}(p, b). \tag{10}$$

Finally, allowing for the possibility that (p, q) may be an interior site which is not distinct from the starting site (a, b) we have the coupled inhomogeneous field equations at $q = b$,

$$6F_{\rm I}(p,b) = 6\delta_{p,a}((a+1) \bmod 2) + [\hat{F}_{\rm I}(p-1,b-1) + F_{\rm I}(p,b-1) + \hat{F}_{\rm I}(p+1,b-1) + \hat{F}_{\rm I}(p+1,b) + F_{\rm II}(p,b+1) + \hat{F}_{\rm I}(p-1,b)] \tag{11}$$

$$6\hat{F}_{\rm I}(p,b) = 6\delta_{p,a}(a \bmod 2) + [F_{\rm I}(p-1,b) + \hat{F}_{\rm I}(p,b-1) + F_{\rm I}(p+1,b) + F_{\rm II}(p+1,b+1) + \hat{F}_{\rm II}(p,b+1) + F_{\rm II}(p-1,b+1)]. \tag{12}$$

In section 3 we present the solutions to equations (1)–(12).

The probabilities for the random walking particle to be absorbed at any specified point on one of the four boundaries, $p = 0$, $p = m+1$, $q = 0$, $q = n+1$, are readily obtained by averaging over the nearest neighbour expectation values subject to the boundary conditions, equations (3)–(8). For example,

$$P(2k-1,0) = \tfrac{1}{6}[\hat{F}_{\rm I}(2k-1,1) + F_{\rm I}(2k-2,1) + F_{\rm I}(2k,1)] \qquad k = 1,\ldots,(m+1)/2. \tag{13}$$

3. Solution

3.1. Homogeneous equations—separation of variables

We begin by solving the homogeneous equations, equations (1), (2), subject to the absorbing boundary conditions, equations (3), (4). The homogeneous field equations separate, on using

$$F(p,q) = P(p)Q(q) \tag{14}$$

$$\hat{F}(p,q) = \hat{P}(p)\hat{Q}(q) \tag{15}$$

into

$$\frac{6Q(q) - Q(q+1) - Q(q-1)}{\hat{Q}(q-1) + \hat{Q}(q)} = \frac{\hat{P}(p-1) + \hat{P}(p+1)}{P(p)} = \lambda \tag{16}$$

$$\frac{6\hat{Q}(q) - \hat{Q}(q+1) - \hat{Q}(q-1)}{Q(q+1) + Q(q)} = \frac{P(p-1) + P(p+1)}{\hat{P}(p)} = \kappa \tag{17}$$

where κ and λ are separation constants. The two coupled equations for P and $\hat{P}$ can be readily decoupled into separate equations

$$P(2k+2) + (2-\lambda\kappa)P(2k) + P(2k-2) = 0 \tag{18}$$

$$\hat{P}(2k+3) + (2-\lambda\kappa)\hat{P}(2k+1) + \hat{P}(2k-1) = 0 \tag{19}$$

where k is an integer. This second-order linear difference equation for $P(\hat{P})$ on a lattice of even (odd) integers has the solution

$$P(2k) = A\mu^k + B\mu^{-k}$$

where

$$\mu = \frac{-(2-\lambda\kappa) + \sqrt{(2-\lambda\kappa)^2 - 4}}{2}$$

and A and B are arbitrary. It follows that the solution of $P(p)$ (with p even) corresponding to the boundary conditions, equations (3), (4), is

$$P(p) = c\sin\left(\frac{\pi jp}{m+1}\right) \tag{20}$$

with

$$\lambda\kappa = 4\cos^2\left(\frac{\pi j}{m+1}\right). \tag{21}$$

The solution for $\hat{P}(p)$ (with p odd) now follows from equation (17) as

$$\hat{P}(p) = \frac{2c}{\kappa}\cos\left(\frac{\pi j}{m+1}\right)\sin\left(\frac{\pi jp}{m+1}\right). \tag{22}$$

Equations (16) and (17) can also be decoupled for $Q(q)$ and $\hat{Q}(q)$ resulting in the same fourth-order linear difference equation in each case:

$$Q(q+4) - (12+\lambda\kappa)Q(q+3) + (38-2\lambda\kappa)Q(q+2) - (12+\lambda\kappa)Q(q+1) + Q(q) = 0. \tag{23}$$

The zeros of the corresponding characteristic quartic polynomial are in reciprocal pairs leading to the solutions

$$Q(q) = c_1\,\mathrm{e}^{\alpha q} + c_2\,\mathrm{e}^{-\alpha q} + c_3\,\mathrm{e}^{\beta q} + c_4\,\mathrm{e}^{-\beta q} \tag{24}$$

$$\hat{Q}(q) = \hat{c}_1\,\mathrm{e}^{\alpha q} + \hat{c}_2\,\mathrm{e}^{-\alpha q} + \hat{c}_3\,\mathrm{e}^{\beta q} + \hat{c}_4\,\mathrm{e}^{-\beta q} \tag{25}$$

where

$$\cosh\alpha = 3 + \frac{\lambda\kappa}{4} + \frac{1}{4}\sqrt{(\lambda\kappa)^2 + 32\lambda\kappa} \tag{26}$$

$$\cosh\beta = 3 + \frac{\lambda\kappa}{4} - \frac{1}{4}\sqrt{(\lambda\kappa)^2 + 32\lambda\kappa}. \tag{27}$$

The coefficients c_1, c_2, c_3, c_4 may be chosen arbitrarily with the coefficients $\hat{c}_1, \hat{c}_2, \hat{c}_3, \hat{c}_4$ then determined by substituting equations (24), (25) into equation (16) or equation (17). From equation (17) we obtain

$$\hat{c}_1 = c_1 f(\alpha) \qquad \hat{c}_2 = c_2 f(-\alpha) \qquad \hat{c}_3 = c_3 f(\beta) \qquad \hat{c}_4 = c_4 f(-\beta) \tag{28}$$

where

$$f(\omega) = \frac{\kappa(\mathrm{e}^{\omega}+1)}{2(3-\cosh\omega)}. \tag{29}$$

By combining equations (14), (15), (20), (22), (24), (25), (28), (29) we can write the solutions to the field equations, equations (1), (2), that satisfy the boundary conditions, equations (3), (4), in the form

$$F(p,q;A,B,C,D) = \sum_{j=1}^{m}\sin\left(\frac{\pi jp}{m+1}\right)[A_j\,\mathrm{e}^{\alpha q}(3-\cosh\alpha) + B_j\,\mathrm{e}^{-\alpha q}(3-\cosh\alpha)$$
$$+ C_j\,\mathrm{e}^{\beta q}(3-\cosh\beta) + D_j\,\mathrm{e}^{-\beta q}(3-\cosh\beta)] \tag{30}$$

$$\hat{F}(p,q;A,B,C,D) = \sum_{j=1}^{m}\sin\left(\frac{\pi jp}{m+1}\right)\cos\left(\frac{\pi j}{m+1}\right)\big[A_j\,\mathrm{e}^{\alpha q}(\mathrm{e}^{\alpha}+1) + B_j\,\mathrm{e}^{-\alpha q}(\mathrm{e}^{-\alpha}+1)$$
$$+ C_j\,\mathrm{e}^{\beta q}(\mathrm{e}^{\beta}+1) + D_j\,\mathrm{e}^{-\beta q}(\mathrm{e}^{-\beta}+1)\big] \tag{31}$$

where A_j, B_j, C_j, D_j are arbitrary.

Before imposing the remaining absorbing boundary conditions, equations (5)–(8), we represent the solutions in the two regions (I: $q \leqslant b$; II: $q \geqslant b$) by

$$F_{\mathrm{I}}(p,q) = F(p,q;A,B,C,D)$$
$$\hat{F}_{\mathrm{I}}(p,q) = \hat{F}(p,q;A,B,C,D)$$
$$F_{\mathrm{II}}(p,q) = F(p,q;\tilde{A},\tilde{B},\tilde{C},\tilde{D})$$
$$\hat{F}_{\mathrm{II}}(p,q) = \hat{F}(p,q;\tilde{A},\tilde{B},\tilde{C},\tilde{D}).$$

The absorbing boundary conditions, equations (5)–(8), can be used to eliminate C, D, $\tilde{C}$, $\tilde{D}$. After an appropriate re-normalization of the coefficients A, B, $\tilde{A}$, $\tilde{B}$ this yields

$$F_{\mathrm{I}}(p,q) = \sum_{j=1}^{m} \sin\left(\frac{\pi jp}{m+1}\right)[A_{\mathrm{I}}(q,\alpha,\beta)A_j + A_{\mathrm{I}}(q,-\alpha,\beta)B_j] \tag{32}$$

$$\hat{F}_{\mathrm{I}}(p,q) = \sum_{j=1}^{m} \sin\left(\frac{\pi jp}{m+1}\right)\cos\left(\frac{\pi j}{m+1}\right)[\hat{A}_{\mathrm{I}}(q,\alpha,\beta)A_j + \hat{A}_{\mathrm{I}}(q,-\alpha,\beta)B_j] \tag{33}$$

$$F_{\mathrm{II}}(p,q) = \sum_{j=1}^{m} \sin\left(\frac{\pi jp}{m+1}\right)[A_{\mathrm{II}}(q,\alpha,\beta)\,\tilde{A}_j + A_{\mathrm{II}}(q,-\alpha,\beta)\tilde{B}_j] \tag{34}$$

$$\hat{F}_{\mathrm{II}}(p,q) = \sum_{j=1}^{m} \sin\left(\frac{\pi jp}{m+1}\right)\cos\left(\frac{\pi j}{m+1}\right)[\hat{A}_{\mathrm{II}}(q,\alpha,\beta)\tilde{A}_j + \hat{A}_{\mathrm{II}}(q,-\alpha,\beta)\tilde{B}_j] \tag{35}$$

where

$$\begin{aligned} A_{\mathrm{I}}(q,\alpha,\beta) = {} & (3-\cosh\alpha)(3-\cosh\beta)2\sinh\beta\,\mathrm{e}^{\alpha q} - (3-\cosh\beta)\,\gamma\,(-\alpha,-\beta)\,\mathrm{e}^{\beta q} \\ & + (3-\cosh\beta)\gamma(-\alpha,\beta)\,\mathrm{e}^{-\beta q} \end{aligned} \tag{36}$$

$$\begin{aligned} \hat{A}_{\mathrm{I}}(q,\alpha,\beta) = {} & 2\,\mathrm{e}^{\alpha q}(\mathrm{e}^{\alpha}+1)(3-\cosh\beta)\sinh\beta - \gamma(-\alpha,-\beta)(\mathrm{e}^{\beta}+1)\,\mathrm{e}^{\beta q} \\ & + \gamma(-\alpha,\beta)(\mathrm{e}^{-\beta}+1)\,\mathrm{e}^{-\beta q} \end{aligned} \tag{37}$$

$$\begin{aligned} A_{\mathrm{II}}(q,\alpha,\beta) = {} & (3-\cosh\alpha)(3-\cosh\beta)2\sinh\beta\,\mathrm{e}^{\alpha q} - \mathrm{e}^{(\alpha-\beta)(n+1)}\gamma(-\alpha,-\beta) \\ & \times (3-\cosh\beta)\,\mathrm{e}^{\beta q} + \mathrm{e}^{(\alpha+\beta)(n+1)}\gamma(-\alpha,\beta)(3-\cosh\beta)\,\mathrm{e}^{-\beta q} \end{aligned} \tag{38}$$

$$\begin{aligned} \hat{A}_{\mathrm{II}}(q,\alpha,\beta) = {} & (\mathrm{e}^{\alpha}+1)(3-\cosh\beta)2\sinh\beta\,\mathrm{e}^{\alpha q} - \mathrm{e}^{(\alpha-\beta)(n+1)}\gamma(-\alpha,-\beta)(\mathrm{e}^{\beta}+1)\,\mathrm{e}^{\beta q} \\ & + \mathrm{e}^{(\alpha+\beta)(n+1)}\gamma(-\alpha,\beta)(\mathrm{e}^{-\beta}+1)\,\mathrm{e}^{-\beta q} \end{aligned} \tag{39}$$

and the function

$$\gamma(\alpha,\beta) = 4\cosh\alpha - 4\cosh\beta - (3-\cosh\beta)\sinh\alpha - (3-\cosh\alpha)\sinh\beta. \tag{40}$$

The coefficients defined in equation (36) which are shown as implicit functions of α and β are also functions of j via equations (21), (26), (27). Equations (32)–(35), are general solutions to the coupled homogeneous field equations, equations (1), (2), that satisfy all of the absorbing boundary conditions, equations (3)–(8).

3.2. Inhomogeneous equations—matching conditions

By using the two matching conditions, equations (9), (10), the four arbitrary constants, A, B, $\tilde{A}$, $\tilde{B}$, can be reduced to two arbitrary constants, say A, B. The solutions in region II can then be written as

$$F_{\mathrm{II}}(p,q) = \sum_{j=1}^{m} \sin\left(\frac{\pi jp}{m+1}\right)[A^{\star}_{\mathrm{II}}(q,\alpha,\beta)A_j + A^{\star}_{\mathrm{II}}(q,-\alpha,\beta)B_j] \tag{41}$$

$$\hat{F}_{\mathrm{II}}(p,q) = \sum_{j=1}^{m} \sin\left(\frac{\pi jp}{m+1}\right)\cos\left(\frac{\pi j}{m+1}\right)[\hat{A}^{\star}_{\mathrm{II}}(q,\alpha,\beta)A_j + \hat{A}^{\star}_{\mathrm{II}}(q,-\alpha,\beta)B_j] \tag{42}$$

where

$$A^{\star}_{\mathrm{II}}(q,\alpha,\beta) = A_{\mathrm{II}}(q,\alpha,\beta)\Gamma_1 + A_{\mathrm{II}}(q,-\alpha,\beta)\Gamma_2 \tag{43}$$

$$\hat{A}^{\star}_{\mathrm{II}}(q,\alpha,\beta) = \hat{A}_{\mathrm{II}}(q,\alpha,\beta)\Gamma_1 + \hat{A}_{\mathrm{II}}(q,-\alpha,\beta)\Gamma_2 \tag{44}$$

and

$$\Gamma_1 = \frac{\hat{A}_{\mathrm{II}}(b,-\alpha,\beta)A_{\mathrm{I}}(b,\alpha,\beta) - A_{\mathrm{II}}(b,-\alpha,\beta)\hat{A}_{\mathrm{I}}(b,\alpha,\beta)}{A_{\mathrm{II}}(b,\alpha,\beta)\hat{A}_{\mathrm{II}}(b,-\alpha,\beta) - \hat{A}_{\mathrm{II}}(b,\alpha,\beta)A_{\mathrm{II}}(b,-\alpha,\beta)} \tag{45}$$

$$\Gamma_2 = \frac{\hat{A}_{\mathrm{II}}(b,\alpha,\beta)A_{\mathrm{I}}(b,\alpha,\beta) - A_{\mathrm{II}}(b,\alpha,\beta)\hat{A}_{\mathrm{I}}(b,\alpha,\beta)}{A_{\mathrm{II}}(b,-\alpha,\beta)\hat{A}_{\mathrm{II}}(b,\alpha,\beta) - A_{\mathrm{II}}(b,\alpha,\beta)\hat{A}_{\mathrm{II}}(b,-\alpha,\beta)}. \tag{46}$$

Finally the remaining arbitrary constants A, B are determined from the requirement that the solutions satisfy the coupled inhomogeneous field equations, equations (11), (12). This step is facilitated using the identity

$$\delta_{p,a} = \frac{2}{m+1}\sum_{j=1}^{m} \sin\left(\frac{\pi ja}{m+1}\right)\sin\left(\frac{\pi jp}{m+1}\right). \tag{47}$$

Explicitly we find:

(i) a even,

$$A_j = \frac{T_1(j)}{T_2(j)}B_j \tag{48}$$

$$B_j = \frac{12}{m+1}\sin\left(\frac{\pi ja}{m+1}\right)\frac{T_2(j)}{T_1(j)T_3(j) - T_4(j)T_2(j)} \tag{49}$$

(ii) a odd,

$$A_j = \frac{T_4(j)}{T_3(j)}B_j \tag{50}$$

$$B_j = \frac{12}{m+1}\sin\left(\frac{\pi ja}{m+1}\right)\frac{T_3(j)}{T_2(j)T_4(j) - T_1(j)T_3(j)} \tag{51}$$

and

$$\begin{aligned} T_1(j) = {} & 2A_{\mathrm{I}}(b,-\alpha,\beta) + 2A^{\star}_{\mathrm{II}}(b+1,-\alpha,\beta) + \hat{A}_{\mathrm{I}}(b-1,-\alpha,\beta) \\ & + \hat{A}^{\star}_{\mathrm{II}}(b+1,-\alpha,\beta) - 6\hat{A}_{\mathrm{I}}(b,-\alpha,\beta) \end{aligned} \tag{52}$$

$$\begin{aligned} T_2(j) = {} & 6\hat{A}_{\mathrm{I}}(b,\alpha,\beta) - 2A_{\mathrm{I}}(b,\alpha,\beta) - 2A^{\star}_{\mathrm{II}}(b+1,\alpha,\beta) \\ & - \hat{A}_{\mathrm{I}}(b-1,\alpha,\beta) - \hat{A}^{\star}_{\mathrm{II}}(b+1,\alpha,\beta) \end{aligned} \tag{53}$$

$$\begin{aligned} T_3(j) = {} & 6A_{\mathrm{I}}(j,b) - 2\cos^2\left(\frac{\pi j}{m+1}\right)\hat{A}_{\mathrm{I}}(b-1,\alpha,\beta) - 2\cos^2\left(\frac{\pi j}{m+1}\right)\hat{A}_{\mathrm{I}}(b,\alpha,\beta) \\ & - A_{\mathrm{I}}(b-1,\alpha,\beta) - A^{\star}_{\mathrm{II}}(b+1,\alpha,\beta) \end{aligned} \tag{54}$$

$$\begin{aligned} T_4(j) = {} & 2\cos^2\left(\frac{\pi j}{m+1}\right)\hat{A}_{\mathrm{I}}(b-1,-\alpha,\beta) + 2\cos^2\left(\frac{\pi j}{m+1}\right)\hat{A}_{\mathrm{I}}(b,-\alpha,\beta) \\ & + A_{\mathrm{I}}(b-1,-\alpha,\beta) + A^{\star}_{\mathrm{II}}(b+1,-\alpha,\beta) - 6A_{\mathrm{I}}(b,-\alpha,\beta). \end{aligned} \tag{55}$$

Table 1. Absorption probabilities at the boundaries on a triangular lattice with a source point enclosed by zig-zag lattice boundaries: (a) exact results using the formulae derived in this paper, (b) results using equations (1), (7*a*)–(7*c*) derived in Keberle and Montet [3] for the straight line boundary approximation.

(a)		(b)	
$P(0, 1)$	0.012 247	$P(2, 0)$	0.019 609
$P(0, 2)$	0.035 646	$P(4, 0)$	0.039 949
$P(0, 3)$	0.054 827	$P(6, 0)$	0.057 904
$P(0, 4)$	0.063 296	$P(8, 0)$	0.065 834
$P(0, 5)$	0.056 686	$P(10, 0)$	0.059 209
$P(0, 6)$	0.039 811	$P(12, 0)$	0.042 817
$P(0, 7)$	0.020 737	$P(14, 0)$	0.024 583
$P(0, 8)$	0.005 743	$P(16, 0)$	0.007 903
$P(1, 0)$	0.026 040	$P(1, 1)$	0.033 624
$P(2, 0)$	0.013 797	$P(0, 2)$	0.036 214
$P(3, 0)$	0.069 523	$P(1, 3)$	0.088 151
$P(4, 0)$	0.021 476	$P(0, 4)$	0.053 888
$P(1, 8)$	0.005 743	$P(17, 1)$	0.015 276
$P(2, 8)$	0.040 253	$P(16, 2)$	0.051 612
$P(3, 8)$	0.015 039	$P(17, 3)$	0.038 562
$P(4, 8)$	0.059 725	$P(16, 4)$	0.075 927

Our solution for the triangular lattice site expectation values with absorbing boundary conditions and a source point is finally given by equations (32), (33) in region I and equations (41), (42) in region II. The relevant quantities appearing in these equations are defined through the series of equations (21), (26), (27), (36)–(40), (43)–(46), (48)–(55).

4. Example

Consider the case of a triangular lattice with $m = 7, n = 7$ and a source at $a = 4, b = 4$ (figure 2). In the approximation to this problem using straight edge boundaries and the (p, q) coordinate system of Keberle and Montet [3] this problem corresponds to $m = 16, n = 7$, $a = 8, b = 4$.

We have calculated expectation values at the nearest neighbour lattice sites around the source and absorption probabilities at the boundaries: (a) using our exact results above, and (b) using the results of Keberle and Montet [3] for the approximation to the problem—their equations (7*a*)–(7*c*). In this small model lattice system we found that the straight line boundary approximation provides reasonable results for expectation values near the source (accurate to within a few per cent) but provides poor results for the absorption probabilities (table 1). Note that the 'absorption probabilities' for the zig-zag boundary coordinates which are inside the straight line boundary of Keberle and Montet [3] are not true probabilities in their solution.

5. Conclusion

Although our solution is somewhat unwieldy, we have nevertheless solved the underlying field equations for random walks on a finite triangular lattice with a single interior source point and zig-zag absorbing boundaries and have thus calculated the associated absorption probabilities. This problem was previously considered to be intractable [3]. We hope that our result will inspire further work in this area.

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